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CALCULATION OF CHARACTERISTICS OF QUEUING SYSTEMS USING THE ERLANG METHOD AND CONSERVATION LAWS

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Abstract. *Any human activity (economic, entrepreneurial, commercial) is associated with the performance of many operations at the stages of movement of raw materials, goods, products from the supply sector to the sphere of production and consumption. Such operations are transportation, storage, processing, sales, etc. These types of activities are characterized by the massive receipt of products, goods, money, and clients at random times; their sequential servicing is carried out by corresponding operations, the execution time of which, as a rule, is also random. All this leads to inequalities in work, creates idle time, underload and overload in operations. Therefore, the tasks arise of analyzing existing options for performing a certain set of operations, identifying bottlenecks and reserves for developing and making management decisions to improve the operating efficiency of any organization. Such problems are successfully solved using queuing theory. The basis for making management decisions are parameters that characterize different aspects of the operation of both the system as a whole and its individual elements. To describe individual parts of the system, the distribution of random variables and their numerical properties are traditionally used. The main characteristics of the action and state of the QS are the average number of requests in the queue or system, the average waiting time for service and others, as well as the value of some probabilities (the probability of service denial, the probability that there are at least a certain number of requirements in the system, the probability that the system is free from maintenance, etc.).*

The article describes some methods and techniques for studying Markov systems (the processes of which have no history), which turn out to be applicable to more general systems. A Markov queuing system with a restriction on the largest number of requirements in the system is considered. The probabilities of state transitions for closed and open-loop systems are described and calculated using Erlang methods and using the laws of conservation of system probabilities and the Laplace transform. Calculations of queue safety under stationary operating modes of queuing systems are demonstrated.

Key words: *queuing systems, Markov systems, Erlang method, closed-loop system, open-loop system, conservation laws.*

РОЗРАХУНОК ХАРАКТЕРИСТИК СИСТЕМ МАСОВОГО ОБСЛУГОВУВАННЯ ІЗ ЗАСТОСУВАННЯМ МЕТОДУ ЕРЛАНГА І ЗАКОНІВ ЗБЕРЕЖЕННЯ

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Анотація. *Будь-яка діяльність людини (господарська, підприємницька, комерційна) пов'язана з виконанням безлічі операцій на етапах руху сировини, товарів, виробів зі сфери постачання до сфери виробництва та споживання. Такими операціями є транспортування, зберігання, обробка, реалізація тощо. Ці види діяльності характеризуються масовістю надходження виробів, товарів, грошей, клієнтів у випадкові моменти часу, їх послідовне обслуговування здійснюється відповідними операціями, час виконання яких здебільшого теж має випадковий характер. Усе це призводить до нерівностей у роботі, породжує простої, недовантаження та перевантаження в операціях. Тому виникають задачі аналізу існуючих варіантів виконання деякої сукупності операцій, виявлення вузьких місць і резервів для розробки та прийняття управлінських рішень щодо підвищення ефективності функціонування будь-якої організації. Такі задачі успішно розв'язуються за допомогою теорії масового обслуговування. Основою для прийняття управлінських рішень є параметри, які характеризують різні сторони дії як системи загалом , так і окремих її елементів. Для опису окремих елементів системи зазвичай використовують розподіл випадкових величин та їх числові характеристики. Основними характеристиками дії і стану СМО є середнє число вимог у черзі або в системі, середній час очікування обслуговування й інші, а також значення деяких імовірностей (імовірності відмови в обслуговуванні, ймовірності того, що в системі розміщено не менше за певне число вимог, імовірності того, що система вільна від обслуговування тощо).*

У статті описані деякі методи і прийоми дослідження марковських систем (процеси яких не мають передісторії), що виявляються застосованими до більш загальних систем. Розглядається марковська система масового обслуговування з обмеженням максимальної кількості вимог у системі. Описані й розраховані ймовірності переходів станів для замкнених і розімкнених систем методами Ерланга та з використанням законів збереження ймовірностей систем і перетворення Лапласа. Продемонстровані обчислення збереження черги за стаціонарних режимів роботи систем масового обслуговування.

Ключові слова: *системи масового обслуговування, марковські системи, метод Ерланга, замкнена система, розімкнена система, закони збереження.*

Introduction

When researching operations, one often has to deal with the operation of peculiar systems called queuing systems (QS). Examples of such systems include: electronic queues, repair shops, ticket offices, queues for family doctors, shops, hairdressers, computer networks.

The QS may consist of a certain number of service units (devices), which are called service channels. Channels can be: communication lines, workplaces, cashiers, salespeople, elevators, cars, etc. QS can be single-channel or multi-channel.

The QS is designed to service the flow of applications (requirements) arriving at its input at random moments in time. Servicing of the request continues for a random time T_s, after which the channel is freed and ready to receive the next request. The random nature of the flow of applications and service times leads to the fact that a large number of applications can accumulate at the input of the QS (they either queue up or leave the QS unserved); in other periods, the QS will work with underload or be completely idle [1].

The QS operation process is a random process with discrete states and continuous time; the state of the QS changes abruptly at the moments of occurrence of events (the arrival of a new request, the end of service, or the moment when an application that is tired of waiting leaves the queue, applications with a limited waiting time).

The subject of the theory of queuing systems is the construction of mathematical models that connect the given operating conditions of the QS (the number of channels, their productivity, operating rules, the nature of the flow of requests) with the performance indicators of interest to the QS, which can be divided into two groups: indicators directly related to the stationary probability distribution $\{P_k\}$ number of applications in the system and time indicators.

The theory of queuing began with the work of the Danish scientist A.K. Erlang for calculating telephone networks. In the formation and development of this theory, a prominent role belongs to scientists A.A. Markov, A.Ya. Khinchin, A.N. Kolmogorov, B.V. Gnedenko, I.N. Kovalenko and many others [2, 3].

Despite the fact that methods have now been developed and classes of QS that are much more general than the class of Markov QS have been quite fully studied, a separate study of the latter is important. This is explained, on the one hand, by the fact that some methods and techniques for studying Markov systems turn out to be applicable (in a modified form) to more general systems and, on the other hand, by the simplicity and clarity of formulas expressing the characteristics of a system through its parameters. This allows us to better understand the qualitative behavior of service processes and evaluate the impact of changing system parameters on the characteristics of interest [2, 4].

Research

Calculation of Markov systems. If the future behavior of a process depends on its past history, then by appropriately expanding the concept of "state," in other words, by increasing the number of components of the vector representing the process, it can be reduced to a Markovian one.

Distribution of intervals for the simplest incoming stream $A(\theta)=1-e^{-\lambda\theta}$, exponential distribution of service time $B(\theta)=1-e^{-\mu\tau}$.

Erlang method. Let's consider a Markov queuing system with a limit on the maximum number of requests in the system by value R. The states of such a system can be characterized by the number of requirements of applications. Possible transitions between states are indicated in Figure 1, and the corresponding probabilities are in Table 1. There is no maintenance in state E_0 (system idle). Applications that find the system in the E_R state are rejected [2, 3].

The probabilities of making two or more jumps are of the order of smallness, higher than Δt, and are not included in the table.

Table 1
Probabilities of transitions in a Markov system over an interval Δt

State of the system		Transition probability accurate to 0 (Δt)
initial	**final**	
$E_0(t)$	$E_1(t+\Delta t)$	$\lambda_0 \Delta t$
$E_0(t)$	$E_0(t+\Delta t)$	$1-\lambda_0 \Delta t$
$E_k(t), k=\overline{1, R-1}$	$E_{k-1}(t+\Delta t)$ $E_{k-1}(t+\Delta t)$ $E_k(t+\Delta t)$	$\mu_k \Delta t$ $\lambda_k \Delta t$ $1-(\lambda_k+\mu_k)\Delta t$
$E_R(t)$	$E_{R-1}(t+\Delta t)$ $E_{R-1}(t+\Delta t)$	$\mu_R \Delta t$ $1-\mu_R \Delta t$

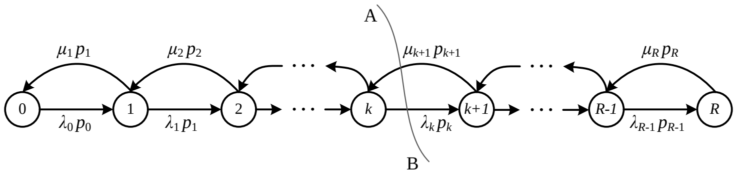

Fig. 1. Law of conservation of state probabilities for Markov systems

Combining the probabilities of events, according to the total probability theorem, we obtain a system of difference equations

$$\begin{cases} p_0(t+\Delta t) = p_0(t)(1-\lambda_0 \Delta t) + \mu_1 \Delta t p_1(t) + 0(\Delta t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_k(t+\Delta t) = p_k(t)[1-(\lambda_k+\mu_k)\Delta t] + \lambda_{k-1}\Delta t p_{k-1}(t) + \\ \quad +\mu_{k+1}\Delta t p_{k+1}(t) + 0(\Delta t), \qquad k=\overline{1, R-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_R(t+\Delta t) = p_R(t)(1-\mu_R \Delta t) + \lambda_{R-1}\Delta t p_{R-1}(t) + 0(\Delta t) \end{cases} \tag{1}$$

Let's regroup the terms of these equations and divide both sides of each of them by Δt:

$$\begin{cases} \dfrac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda_0 p_0(t) + \mu_1 p_1(t) + \dfrac{0(\Delta t)}{\Delta t} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{p_k(t+\Delta t) - p_k(t)}{\Delta t} = -(\lambda_k+\mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \\ \quad +\mu_{k+1}p_{k+1}(t) + \dfrac{0(\Delta t)}{\Delta t}, \qquad k=\overline{1, R-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{p_R(t+\Delta t) - p_R(t)}{\Delta t} = -\mu_R p_R(t) + \lambda_{R-1}p_{R-1}(t) + \dfrac{0(\Delta t)}{\Delta t} \end{cases} \tag{2}$$

Letting Δt tend to zero, we obtain a system of differential equations

$$\begin{cases} p_0^{'}(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_k^{'}(t) = -(\lambda_k+\mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \\ \quad +\mu_{k+1}p_{k+1}(t), \; k=\overline{1, R-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_R^{'}(t) = -\mu_R p_R(t) + \lambda_{R-1}p_{R-1}(t) \end{cases} \tag{3}$$

By specifying the initial conditions for system (3), for example, in the form $p_0(0)=1$ and $p_k(0)=0$ for $k=\overline{1,R}$, one can find a numerical solution to the corresponding Cauchy problem for an arbitrary value of *t*.

As $t\to\infty$, the time derivatives, if a stationary regime exists, vanish. As a result, we arrive at a system of linear algebraic equations for stationary probabilities

$$\begin{cases} -\lambda_0 p_0 + \mu_1 p_1 = 0 \\ \lambda_0 p_0 - (\lambda_1+\mu_1)p_1 + \mu_2 p_2 = 0 \\ \lambda_1 p_1 - (\lambda_2+\mu_2)p_2 + \mu_3 p_3 = 0 \\ \dots\dots\dots\dots\dots\dots \\ \lambda_{k-1}p_{k-1} - (\lambda_k+\mu_k)p_k + \mu_{k+1}p_{k+1} = 0 \\ \dots\dots\dots\dots\dots\dots \\ \lambda_{R-1}p_{R-1} - \mu_R p_R = 0 \end{cases} \quad (4)$$

Adding the first *k* + 1 equations of this system, we are convinced that

$$-\lambda_k p_k + \lambda_{k+1}p_{k+1} = 0 \quad (5)$$

whence it follows

$$p_{k+1} = \frac{\lambda_k}{\mu_{k+1}}p_k = \frac{\lambda_k}{\mu_{k+1}}\frac{\lambda_{k-1}}{\mu_k}p_{k-1} = \ldots = \frac{\lambda_k}{\mu_{k+1}}\frac{\lambda_{k-1}}{\mu_k}\ldots\frac{\lambda_2}{\mu_2}p_1. \quad (6)$$

From the first equation we get $p_0=\frac{\lambda_0}{\mu_0}p_0$. Thus, for a problem with transitions only to neighboring states (the "reproduction and death" scheme), all probabilities are expressed through p_0 according to

$$p_k = \alpha_k p_0,\ k=\overline{1,R},$$

where $\alpha_k = \prod_{i=0}^{k-1}\frac{\lambda_i}{\mu_{i+1}}$.

The probability p_0 is determined from the normalization condition (the sum of all probabilities must be equal to one). Consequently,

$$p_0 = \left(1+\sum_{k=1}^{R}\alpha_k\right)^{-1} \quad (7)$$

From the considered scheme, various special cases can be obtained, for example, $\lambda_K=\lambda(R-K)K$ (closed-loop system), $\lambda_K=\lambda$ (open-loop system), as well as the case of an *n*-linear system, for which

$$\mu_k = \begin{cases} k\mu, k=\overline{1,n-1} \\ n\mu,\ k=\overline{1,R} \end{cases}.$$

For an open-loop system, $R=\infty$ corresponds to an unlimited queue. For $R=\infty$, the condition for the existence of a stationary solution is the convergence of the series $\sum_{K=1}^{\infty}\alpha_K$. For the series to converge, it is sufficient to require the existence of a constant $q\in(0,1)$, which exceeds any ratio $\frac{\lambda_K}{\mu_1}p_0$, starting from some number *k*. In this case, subsequent terms of the series are majorized by terms of a decreasing geometric progression with denominator *q*, the sum of which is finite.

The final Erlang formulas for the open-loop system $M/M/n$:

$$\begin{cases} p_0 = \left[1+\sum_{i=1}^{n}\left(\frac{\lambda}{\mu}\right)^i\frac{1}{i!}+\left(\frac{\lambda}{\mu}\right)^n\frac{\lambda}{n!(n\mu-\lambda)}\right]^{-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p_k = p_0\left(\frac{\lambda}{\mu}\right)^k\frac{1}{k!},\ k=\overline{1,n} \\ p_0 = p_n\left(\frac{\lambda}{n\mu}\right)^{k-n},\ k=n+1,n+2,\ldots \end{cases} \quad (8)$$

For systems with a finite number of states, especially closed-loop ones, $p_0=1$, and using formula (5) we calculate $\{p_k\}$ for specific dependencies $\mu(k)$ and $\lambda(k)$, simultaneously calculating the sum, and then normalize the results.

For the abbreviated designation of QS, D. Kendall proposed using a formula of the form $A/B/n/R$, where *A* indicates the distribution of intervals between requests, *B* is the distribution of service time, *n* is the number of channels, R is the maximum number of requests in the system [3, 5].

Calculation of the** M / M / 1* ***system using the laws of conservation of state probabilities. Let us apply the laws of conservation of state probabilities to the calculation of the main indicators of the $M/M/1$ system (single-channel QS with the simplest incoming flow and exponential distribution with the Markov property, with an unlimited queue).

First of all, we note that for this QS $\lambda_K = \lambda$ and $\mu_K = \mu$ and does not depend on *k*. Consequently, for all *K* we have $\mu p_{K+1} = \lambda p_k$, or $p_{K+1} = \frac{\lambda}{\mu} p_k$. By introducing the notation $x = \frac{\lambda}{\mu}$ – system load factor, we obtain formulas for stationary probabilities $p_K = (1-x)x^k, k = 0,1,\dots$

The condition for the existence of a stationary regime is the inequality λ<μ. Note that $\frac{\lambda}{\mu} = \bar{\tau}$ ($\bar{\tau}$ is the average service time). Consequently, $p_0 = 1 - \frac{\lambda}{\mu} = 1 - \lambda\bar{\tau}$, which confirms the law of conservation of requirements $P\left(1 - \frac{s}{\lambda}\right) = \upsilon(s)$.

Generating function of probabilities

$$P(z) = \sum_{k=0}^{\infty} (1-x) x^k z^k = \frac{1-x}{1-xz} \tag{9}$$

Based on the formula, the Laplace transform of the distribution of the application's residence time in the system

$$\upsilon(s) = P\left(1 - \frac{s}{\lambda}\right) = \frac{1-x}{1 - x\left(1 - \frac{s}{\lambda}\right)} = \frac{\mu - \lambda}{\mu - \lambda + s} \tag{10}$$

Consequently, the residence time of a request in the system is subject to the exponential law with the parameter $\lambda - \mu$. Average stay time

$$\upsilon_1 = \frac{1}{\mu - \lambda},$$

as $\lambda \to \mu$ increases without limit. This leads to the conclusion that it is inadmissible to choose an average service speed equal to the intensity of the flow of requests. Average number of applications in the system

$$L_1 = (1-x)\sum_{k=1}^{\infty} kx^k = x(1-x)\sum_{k=1}^{\infty} kx^{k-1} =$$

$$= x(1-x)\frac{d}{dx}\frac{1}{(1-x)} = \frac{x}{(1-x)} = \frac{\lambda}{\mu - \lambda}$$

Thus, the relation *L*[1]=λv$_1$ *υ* between L_1 and v_1 turned out to be correct, which is consistent with the queue conservation law.

Conclusion

The paper considers the possibility of using the Erlang method to construct models of queuing systems with a limit on the number of requests associated with the system.

Calculations of queue conservation under stationary operating modes of the M / M / 1 type QS using conservation laws and Laplace transforms, generating distribution functions are shown.

Література

1. Saaty T.L. Elements of Queueing Theory, with Applications. Published by Dover Publications. 1983, 423 p. ISBN 10: 0486645533 / ISBN 13: 9780486645537.
2. Математичні основи теорії телекомунікаційних систем. / В. В. Поповський, С. О. Сабуров, В. Ф. Олійник, Ю. І. Лосєв, Д. В. Агеєв [та ін.] ; за заг. ред. В. В. Поповського. Харків : ТОВ «Компанія СМІТ», 2006. 564 с. ISBN 966-8530-60-8.
3. Ларіонов Ю. І., Марченко Л. С., Хаджмуратов М. А. Дослідження операцій : навч. посіб. для вузів. Частина 2. Харків : Інжек, 2005. 288 с. ISBN 966-8515-91-9.
4. Димова Г. О., Ларченко О. В. Моделі і методи інтелектуального аналізу даних : навчальний посібник. Херсон : Книжкове видавництво ФОП Вишемирський В. С., 2021. 142 с. ISBN 978-617-7941-60-5.
5. Moor J.H., Weatherford L.R. Decision Modeling With Microsoft Excel. Upper Sddle River, New Jersey 07458: Published by Prentice Hall, Inc. 2001, 693 p. ISBN 0-13-017789-X.

References

1. Saaty, Thomas L. (1983). Elements of Queueing Theory, with Applications. Published by Dover Publications. ISBN 10: 0486645533 / ISBN 13: 9780486645537.

2. Popovs'kyy, V.V.; Saburova, S.O.; Oliynyk, V.F.; Losyev, YU.I.; Aheyev, D.V. & ta in. (2006). Matematychni osnovy teoriyi telekomunikatsiynykh system [Mathematical foundations of the theory of telecommunication systems]. Kharkiv: Kompaniya SMIT. ISBN 966-8530-60-8 [in Ukrainian].

3. Larionov, Yu.I.; Marchenko, L.S. & Hazhmuradov, M.A. (2005). Doslidzhennya operatsiy. Chastyna II: Navchal'nyy posibnyk [Operational Research. Part II: Study Guide]. Kharkiv: ENGEC PH. ISBN 966-8515-91-9 [in Ukrainian].

4. Dymova, H.O. & Larchenko, O.V. (2021). Modeli i metody intelektual'noho analizu danykh: navchal'nyy posibnyk [Models and Methods of Intelligent Data Analysis: Tutorial]. Kherson: Vyshemyrs'kyy V.S. ISBN 978-617-7941-60-5 [in Ukrainian].

5. Moor, Jeffrey H. & Weatherford, Larry R. (2001). Decision Modeling with Microsoft Excel. Upper Sddle River, New Jersey 07458: Published by Prentice Hall, Inc. ISBN 0-13-017789-X.